

Quantum Mechanics as a Classical Theory

IX: The Formation of Operators and Quantum Phase-Space Densities

L. S. F. Olavo

Departamento de Fisica, Universidade de Brasilia - UnB
CEP 70910-900 Brasilia, D.F. - Brazil

February 1, 2008

Abstract

In our previous papers we were interested in making a reconstruction of quantum mechanics according to classical mechanics. In this paper we suspend this program for a while and turn our attention to a theme in the frontier of quantum mechanics itself—that is, the formation of operators. We then investigate all the subtleties involved in forming operators from their classical counterparts. We show, using the formalism of quantum phase-space distributions, that our formation method, which is equivalent to Weyl's rule, gives the correct answer. Since this method implies that eigenstates are not dispersion-free we argue for modifications in the orthodox view. Many properties of the quantum phase-space distributions are also investigated and discussed in the realm of our classical approach. We then strengthen the conclusions of our previous papers that quantum mechanics is merely an extremely good approximation of classical statistical mechanics performed upon the configuration space.

1 The Formation of Quantum Operators

The formation problem of quantum mechanical operators has already been treated by a number of authors [1, 2, 3, 4, 5, 6]. The problem resides basically in forming, from a given classical function $f(q, p)$ of the commuting generalized coordinates and momenta, the related quantum mechanical operator $O[f(p, q)]$.

This problem, as expressed in Shewell's review article [3], either cannot be solved in a non-ambiguous way (e.g. von Neumann's and Dirac's rules) or it does not give the result expected by the orthodox epistemology of quantum mechanics, viz. Weyl's and Revier's rules.

As an example, the operators related to the classical function $f(q, p) = q^2 p^2$ are

$$\text{von Neumann: } \begin{cases} O[q^2 p^2] = O[q]^2 O[p]^2 - 2i\hbar O[q]O[p] - \hbar^2/4 \\ O[q^2 p^2] = O[q]^2 O[p]^2 - 2i\hbar O[q]O[p] - \hbar^2 \end{cases} \quad (1)$$

and

$$\text{Dirac: } \begin{cases} O[q^2 p^2] = O[q]^2 O[p]^2 - 2i\hbar O[q]O[p] - \hbar^2/3 \\ O[q^2 p^2] = O[q]^2 O[p]^2 - 2i\hbar O[q]O[p] - 2\hbar^2/3 \end{cases} \quad (2)$$

which are ambiguous and so, contrary to the postulates of quantum mechanics—each ‘observable’ must be assigned to one and only one operator.

For Weyl’s rule we have

$$O[q^2 p^2] = O[q]^2 O[p]^2 - 2i\hbar O[q]O[p] - \hbar^2/2 \quad (3)$$

which is unambiguous but has the drawback that, for the harmonic oscillator problem with the hamiltonian given by

$$H = \frac{1}{2}(p^2 + q^2), \quad (4)$$

this rule gives (figure I)

$$O[H^2] = \{O[H]\}^2 + \hbar^2/4 \quad (5)$$

which is equivalent to saying that it predicts energy dispersions for the energy eigenstates (that is also contrary to the orthodox epistemological considerations of quantum mechanics). In this example, equation (3) says that the dispersion will be given by

$$(\Delta E)^2 = \langle O[H^2] \rangle - \langle O[H] \rangle^2 = \hbar^2/4, \quad (6)$$

for all energy levels.

Weyl’s rule has also the disadvantage that some functions of classical constants of motion are transformed into operators which do not commute with the hamiltonian and are not constants of motion in the quantum mechanical side. This may be shown using the hamiltonian[3]

$$H = p^2/2 + q^4/4, \quad (7)$$

for which, Weyl’s rule gives

$$O[H^2] = \{O[H]\}^2 + 3\hbar^2 q^2/4, \quad (8)$$

implying that $O[H^2]$ does not represent a constant of motion, since it does not commute with H .

In short, the acceptance of Weyl’s rule implies that we do not accept the first von Neumann’s rule [3]

$$\text{if } O[A] = \mathbf{A} \Rightarrow O[f(A)] = f(\mathbf{A}). \quad (9)$$

It is possible to show that the symmetrization rules may be reduced to Weyl's rule and present the same problems.

It is the aim of the next section to show, using our formalism [7, 8, 9, 10, 11, 12, 13, 14] why ambiguities arrive and to throw some confidence upon Weyl's rule, which is equivalent to our own.

In the third section, we show that this result is in accordance with the results obtained using the classical phase-space functions derived using our previously defined formalism [7, 8, 9, 10, 11, 12, 13, 14].

Some of the main characteristics of these functions will be dealt with in the fourth section by means of some illustrative examples that will throw some light upon the discussion.

2 The Ambiguity Problem

It has been already shown that the ambiguity related to von Neumann's and Dirac's rules (to cite but a few) are related with the attempt of mapping a commutative ring into a non-commutative one [2]. We will show this using our analytical method.

From our previous work [7] one might see that the Schrödinger equation of quantum mechanics is derived from the classical Liouville equation using the Infinitesimal Wigner-Moyal Transformation

$$\rho\left(q - \frac{\delta q}{2}, q + \frac{\delta q}{2}; t\right) = \int F(q, p; t) e^{\frac{i}{\hbar} p \delta q} dp, \quad (10)$$

where $F(q, p; t)$ is the classical phase-space probability density function and $\rho(q, \delta q; t)$ is the quantum density function (also called density matrix) and where δq is an infinitesimal increment *taken over the coordinate q -axis*.

To derive Schrödinger equation it is necessary to impose upon the density function the functional form

$$\rho\left(q - \frac{\delta q}{2}, q + \frac{\delta q}{2}; t\right) = \psi^\dagger\left(q - \frac{\delta q}{2}; t\right) \psi\left(q + \frac{\delta q}{2}; t\right) \quad (11)$$

and to expand this product with

$$\psi(q; t) = R(q; t) e^{iS(q; t)/\hbar} \quad (12)$$

in powers of the infinitesimal increment δq . We then take the density (11) into its equation and, since this increment is infinitesimal, we keep only the zeroth and first order resulting terms and show that the former implies the continuity equation while the later implies Schrödinger equation.

It is noteworthy that in this procedure we have lost one of the independent variables, viz. the infinitesimal δq related to p . This makes our problem much simpler but it has the disadvantage of introducing the non-commutativity of

the canonically conjugated quantities. The explanation why this formalism is rather triumphant is related with the fact that the error introduced with non-commutativity is of the order of Planck's constant and so, the method is expected to be efficient to within this approximation (which is rather good).

The construction of operators is a direct result of the definition (10). Since the mean values of a classical function $f(q, p)$ are given by

$$\langle f(q, p) \rangle = \int \int f(q, p) F(q, p; t) dq dp, \quad (13)$$

it is automatic that the mean of the operator related with this function is given by[7]

$$\langle O[f(q, p)] \rangle = \int \lim_{\delta q \rightarrow 0} f\left(\hat{q}, -i\hbar \frac{\partial}{\partial(\delta q)}\right) \rho\left(q - \frac{\delta q}{2}, q + \frac{\delta q}{2}; t\right) dx \quad (14)$$

giving the same values of expression (13).

The function $f\left(\hat{q}, -i\hbar \frac{\partial}{\partial(\delta q)}\right)$ is unambiguous defined since q and δq are independent variables.

Now, giving the amplitudes $\psi(q; t)$, we can form $\rho\left(q - \frac{\delta q}{2}, q + \frac{\delta q}{2}; t\right)$ according to expression (11) and expand these amplitudes in a Taylor series around q to get, until second order in δq

$$\begin{aligned} \langle O[f(q, p)] \rangle = & \int \lim_{\delta q \rightarrow 0} f\left(\hat{q}, -i\hbar \frac{\partial}{\partial(\delta q)}\right) \left[\psi^\dagger \psi + \left(\frac{\delta q}{2}\right) \left(\psi^\dagger \frac{\partial \psi}{\partial q} - \frac{\partial \psi^\dagger}{\partial q} \psi \right) + \right. \\ & \left. + \frac{1}{2!} \left(\frac{\delta q}{2}\right)^2 \left(\psi^\dagger \frac{\partial^2 \psi}{\partial q^2} + \frac{\partial^2 \psi^\dagger}{\partial q^2} \psi - 2 \frac{\partial \psi}{\partial q} \frac{\partial \psi^\dagger}{\partial q} \right) + o(\delta q^3) \right], \end{aligned} \quad (15)$$

and then apply to each term of the expansion the operator \hat{f} defined in (14) and take the limit as indicated.

The procedure defined above is unambiguous and, it might be easily shown, gives the same results of Weyl's rule. We have, for example, in the case of $f(q, p) = q^2 p^2$, the operator

$$\hat{f}\left(\hat{q}, -i\hbar \frac{\partial}{\partial(\delta q)}\right) = \lim_{\delta q \rightarrow 0} \hat{q}^2 \left(-\hbar^2 \frac{\partial^2}{\partial(\delta q)^2} \right) \quad (16)$$

giving, in terms of the amplitudes, the operator

$$\hat{f} = \hat{q}^2 \hat{p}^2 - 2i\hbar \hat{q} \hat{p} - \hbar^2/2. \quad (17)$$

However, in expression (14), we could have considered the term q^2 as part of the function $\rho\left(q - \frac{\delta q}{2}, q + \frac{\delta q}{2}; t\right)$; in the example above this means that we are calculating

$$\langle O[f(q, p)] \rangle = \int \lim_{\delta q \rightarrow 0} \left(-\hbar^2 \frac{\partial^2}{\partial(\delta q)^2} \right) \left[q^2 \rho\left(q - \frac{\delta q}{2}, q + \frac{\delta q}{2}; t\right) \right] dq, \quad (18)$$

which is allowed since q and δq are independent variables. In this case we will expand the function

$$q^2 \rho \left(q - \frac{\delta q}{2}, q + \frac{\delta q}{2}; t \right) = q^2 \psi^\dagger \left(q - \frac{\delta q}{2}; t \right) \psi \left(q + \frac{\delta q}{2}; t \right) \quad (19)$$

and there are three possibilities of doing that

$$\begin{cases} \{q^2 \psi^\dagger \left(q - \frac{\delta q}{2}; t \right)\} \psi \left(q + \frac{\delta q}{2}; t \right) \mapsto \hat{q}^2 \hat{p}^2 - 2i\hbar \hat{q} \hat{p} - \hbar^2 \\ \psi^\dagger \left(q - \frac{\delta q}{2}; t \right) \{q^2 \psi \left(q - \frac{\delta q}{2}; t \right)\} \mapsto \hat{q}^2 \hat{p}^2 - 2i\hbar \hat{q} \hat{p} - \hbar^2 \\ \{q \psi^\dagger \left(q - \frac{\delta q}{2}; t \right)\} \{q \psi \left(q - \frac{\delta q}{2}; t \right)\} \mapsto \hat{q}^2 \hat{p}^2 - 2i\hbar \hat{q} \hat{p} \end{cases} \quad (20)$$

or we could make the expansion of

$$q \left(-\hbar^2 \frac{\partial^2}{\partial(\delta q)^2} \right) \left[q \psi^\dagger \left(q - \frac{\delta q}{2}; t \right) \psi \left(q + \frac{\delta q}{2}; t \right) \right] \quad (21)$$

giving two other possibilities.

All these results are different from the previous one (17) because the infinitesimal increment is taken over the coordinate q -axis and taking the limit $\delta q \rightarrow 0$ implies making one variable to disappear.

Another way to see that is making the coordinate transformation

$$y = q + \frac{\delta q}{2} ; y' = q - \frac{\delta q}{2} \quad (22)$$

giving

$$\frac{\partial}{\partial(\delta q)} = \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'} \right) ; \rho = \rho(y', y; t) \quad (23)$$

We then have

$$\rho(y', y; t) = \sum_{n=0}^{\infty} \frac{1}{n!} \left[(y - y_0) \frac{\partial}{\partial y} + (y' - y'_0) \frac{\partial}{\partial y'} \right]^n \rho(y'_0, y_0; t) \quad (24)$$

where it is implicit that we first perform the derivatives and then compute their values at

$$y_0 = y'_0 = q \quad (25)$$

Using expression (23) we get for the mean value of the classical phase-space monomial represented by $q^N p^M$ the corresponding expression

$$\begin{aligned} \langle q^N p^M \rangle &= (-i\hbar)^M \int \frac{(y + y')^N}{2^N} \left[\frac{\partial}{\partial y} - \frac{\partial}{\partial y'} \right]^M \cdot \\ &\cdot \sum_{n=0}^{\infty} \frac{1}{n!} \left[(y - y_0) \frac{\partial}{\partial y} + (y' - y'_0) \frac{\partial}{\partial y'} \right]^n \rho(y'_0, y_0; t) d \left[\frac{y + y'}{2} \right]. \end{aligned} \quad (26)$$

Now, since y' and y are independent coordinates we might change the order of some of the terms in the above expression which are of the form

$$y^k \frac{\partial}{\partial y'^j} ; y'^k \frac{\partial}{\partial y^j} \quad (27)$$

but when we calculate the limit we see that, in this limit,

$$y = y' = q \quad (28)$$

and are not independent quantities anymore (they are, indeed, equal). This limiting process is exactly the one we have used, as described above, to derive the Schrödinger equation and is also, as already shown[7] the one responsible for the introduction of non-commutativity of the dynamic operators.

We are not able to choose, at this place, one or another of the procedures as shown in the examples (19,20) and (21) but the claims for the procedure leading to Weyl's rule are obvious. This is because it is not clear why we will expand that part of the operator (we are trying to find an expression for) as part of the density function; indeed, in expression (16), for example, the term q^2 refers to an *operator* while the density function is indexed by q , a *coordinate* variable.

We must be aware however, of the consequences of this choice. One of them, for example, will be to predict, for the harmonic oscillator example, a dispersion in the energy according to expression (5) of

$$\Delta E = \frac{\hbar}{2}, \quad (29)$$

a result to which we will soon return.

In the next section, we will show that these claims for Weyl's rule is strengthened when we return, by means of expression (10), to the classical phase-space formalism. We will also discuss some of the properties of these functions in the realm of the present developments.

3 Classical Phase-Space Distributions

Distributions of probability defined upon classical phase-space were first proposed by Wigner[15] in connection with thermodynamics considerations. Later, the theme was further developed by Moyal[16] and Groenewold[2]. In the late sixties, the problem began again to deserve some attention mainly in connection with problems on the coherent properties of the electromagnetic field[5], on the theory of laser and on non-linear processes[1, 6].

More recently, we have seen the revival of the problem of writing quantum mechanics using these distributions and the notion of stochastic phase-space [19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29].

All these latter developments are rather technical and pay more attention to formal aspects of these distributions than to the epistemological considerations

that could follow from these developments. Such considerations were indeed well formulated in Moyal's paper where the epistemological deficiencies related to the tentative of linking classical and quantum mechanics by writing the later in classical phase-space were pointed out.

The major problem with the distributions $F(q, p; t)$, derived from expressions nearly similar to (10) and defined over classical phase-space—as recognized by all the cited authors—is the fact that these functions might take negative values; they are then called quasi-probability functions. This fact is usually interpreted as meaning our necessarily introduction of an error when passing from a fundamental theory (quantum mechanics with non-commuting 'observables') to an approximate theory (classical statistical mechanics with commuting dynamic variables).

In the context of the present series of papers this conclusion has to be reversed. We have already shown that it is possible to derive quantum mechanics from classical mechanics without introducing any non-classical postulate. We have also shown that this process of derivation, where one of the phase-space coordinates (usually the momentum) disappears, introduces the non-commutativity when we impose that our densities have to obey equation (11). The hierarchy of the theories is fixed by this derivation and we see that quantum mechanics will give good results whenever the experimental environment is not able to uncover Nature with techniques where the dispersion product is given by

$$\Delta q \Delta p < \frac{\hbar}{2}, \quad (30)$$

where q and p are the generalized canonically conjugate coordinate and momentum. That is to say: *the dispersion relations do not impose any constraint upon the behavior of Nature but only upon our capacity of describing Nature by means of quantum theory.*

Because of the smallness of Planck's constant, we might conclude that this technique is rather fruitful and the error introduced by the quantum formalism is far from being detectable with nowadays laboratory technology.

Then, as we have already shown, the passage from a description based upon the classical density function to a description based upon the density 'matrix' $\rho(q, \delta q)$ is exact but we still have two independent variables and, as in the classical phase-space, no dispersions[7]. When passing from densities $\rho(q, \delta q)$ to a description using the amplitudes $\psi(q; t)$, the dispersion relation becomes

$$\Delta q \Delta p \geq \frac{\hbar}{2}, \quad (31)$$

but the problem becomes rather tractable and, as the error introduced is absurdly small, we might conclude that more was gained than was lost.

Now, when trying to return to classical phase-space—not by inverting the Fourier Transform (10) but by finding the function $F(q, p; t)$ that, when integrated will result in the desired $\rho(q, \delta q)$ [9]—we shall admit that the introduced

error related with the dispersion relation and with the imposition (11) will propagate upward and will necessarily be incorporated into our densities $F(q, p; t)$. This fact might be easily verified from the usual expression[16] for the classical density

$$F(q, p; t) = e^{-\frac{1}{2}i\hbar\partial^2/\partial p\partial q} \left[\psi^\dagger(q)\phi(p)e^{ipq/\hbar} \right], \quad (32)$$

where $\psi(q)$ and $\phi(p)$ are the position and momentum eigenfunctions. This expression does indeed incorporate[16] Heisenberg's inequality.

The fact that such functions are not positive definite might be taken as an indication that the process of quantization—equation (11) and the limit $\delta x \rightarrow 0$ —indeed introduces an error.

Another problem with these distributions is the fact that, although they give the correct values for the energies mean values and the dispersion relations for q and p , they also predict a dispersion in the energy eigenstates—something rather contrary to the orthodox interpretation where eigenstates are dispersionless.

This is the point we wanted to reach that will help us with the discussions about which method of operator construction to choose.

In the next section we will develop some examples of classical phase-space distributions for quantum systems and will study more deeply the harmonic oscillator problem to stress some relevant points for our latter discussion.

4 Some Examples: The Harmonic Oscillator

We begin by giving some obvious examples:

(i) Free Particle:

$$\psi(x, t) = e^{ikx} \Rightarrow \rho(x, \delta x) = e^{ik\delta x} \Rightarrow F(x, p; t) = \delta_D(k - p/\hbar) \quad (33)$$

where $\delta_D(x)$ is the Dirac's delta function. This density $\rho(x, \delta x)$ shall be compared to the one obtained when passing from the classical phase-space to the 'quantum' $(x, \delta x)$ configuration space. Indeed, we have already shown[9] that, for a dispersion free *ensemble* of systems with one free particle, where the classical density function is given by

$$F(x, p; t) = \delta_D[x - x_0(t)]\delta_D[p - p_0(t)], \quad (34)$$

the correct density function is given by

$$\rho(x, \delta x) = \delta_D[x - x_0(t)]e^{ik\delta x} \quad (35)$$

and represents an *ensemble* of systems with one localized free particle. Since this density function represents a dispersion free *ensemble* it cannot be expressed within quantum theory based upon Schrödinger's equation. We thus clearly see that, when passing from the quantum to the classical, this localization cannot

be retrieved and we are left with expression (33) which has no information about the position of the particle—this information was lost.

(ii) Minimum Free Wave-Packet:

$$\psi(x, t) = e^{-x^2/2\alpha^2 + ik_0 x} \Rightarrow \rho(x, \delta x) = e^{-x^2/2\alpha^2} e^{-(\delta x)^2/4\alpha^2 + ik_0 \delta x}, \quad (36)$$

where $a = \Delta x \sqrt{2}$. We then have

$$F(x, p; t) = N e^{-[x^2/2\alpha^2 + (p - \hbar k_0)^2 a^2 / \hbar^2]}, \quad (37)$$

which implies that $\Delta p = \hbar \sqrt{2}/a$ and so

$$\Delta x \Delta p = \hbar/2. \quad (38)$$

We now want to develop another example more deeply.

(iii) The Harmonic Oscillator: The probability amplitudes are given by

$$\psi_n(x) = \left[\frac{1}{2^n n! \pi^{1/2}} \right]^{1/2} H_n(x) e^{-\frac{1}{2}x^2}, \quad (39)$$

where $H_n(x)$ is the n-th Hermite polynomial. These amplitudes are solution of the Schrödinger equation

$$\hat{E} \psi_n(x) = \frac{1}{2} [\hat{p}^2 + \hat{q}^2] \psi_n(x) = E_n \psi_n(x) = (n + 1/2) \hbar \psi_n(x), \quad (40)$$

where \hat{E} is the hamiltonian operator and E_n its eigenvalues.

Using expression (10) we might explicitly calculate $F(x, p; t)$ to get the normalized functions

$$\begin{cases} n = 0 & F_0(x, p; t) = \frac{1}{\pi \hbar} e^{-H} \\ n = 1 & F_1(x, p; t) = \frac{2}{\pi \hbar} \left[H - \frac{1}{2} \right] e^{-H} \\ n = 2 & F_2(x, p; t) = \frac{2}{\pi \hbar} \left[(H - 1)^2 - \frac{1}{2} \right] e^{-H} \\ n = 3 & F_3(x, p; t) = \frac{2}{3\pi \hbar} \left[2(H - 2)^3 + (H - 2)^2 - 3(H - 2) - \frac{7}{2} \right] e^{-H} \end{cases}, \quad (41)$$

where

$$H = x^2 + p^2 / \hbar^2. \quad (42)$$

In figures II, III and IV we plot the choices $n = 0, 3$ and 10 respectively. It is clear that, apart from the ground state, we have the appearing of negative probability densities.

As amply noticed in the literature, we get non-negative densities when we integrate in one or other phase-space coordinates giving the lateral probabilities

$$F_1(x; t), F_2(p; t) \geq 0 \quad \forall x \text{ and } \forall p \quad (43)$$

which are all in accordance with the quantum mechanical results.

With the density functions $F(x, p; t)$ at hand we might calculate the mean energies, the Heisemberg dispersion relations and the mean quadratic deviation (the dispersion) of these energies. We calculated these values for the first $n = 0, \dots, 7$ functions and quoted the results in table I.

Looking at this table we can see that the energy mean values and the Heisemberg dispersion relations are given by

$$\langle E_n \rangle = (n + 1/2)\hbar \text{ and } \Delta x \Delta p = (n + 1/2)\hbar \quad (44)$$

as expected[17]. For the energy mean quadratic deviation we get

$$\Delta E = \frac{\hbar}{2} \quad (45)$$

for all states.

We now return to expression (4) and see that result (45) agrees with the way we define operators using our formalism (and also with Weyl's rules). It is important to stress that, when finding $F(x, p; t)$ *from* the quantum mechanical amplitudes we have not made any approximations—we have just performed the ‘inversion’ indicated by (10). The value of the energy dispersion shall agree with the quantum mechanical one as in the cases of the mean energy and Heisemberg's relations (44).

We then must abandon the first von Neumann's rule[3] represented in expression (9) and also conclude that even eigenstates of the Schrödinger equation are not dispersion-free.

The results above are good indications that our operator construction procedure, and also Weyl's rule, is the correct one to be used in the formal structure of quantum mechanics.

Another consequence of expression (45), which may be written generally as

$$\Delta E = \frac{\hbar}{2}\omega, \quad (46)$$

where ω is the frequency of the harmonic oscillator (we have made equal to unity in the previous development), is the conclusion, based on the Heisemberg's energy-time dispersion relation, that the dispersion in time must be given by

$$\Delta t = \frac{1}{\omega} = \frac{T}{2\pi}, \quad (47)$$

where T is the period of oscillation.

We also note that, considering expressions (44) and (46), the correct energies (within the mean quadratic deviations) of the harmonic oscillator problem shall be written as

$$E_n = \langle E_n \rangle \pm \Delta E = (n + \frac{1}{2})\hbar \pm \frac{\hbar}{2}, \quad (48)$$

which throws some light upon the meaning of the vacuum energy fluctuation (we will return to this problem in a latter work).

5 Constants of Motion

The acceptance of our rules of operator formation leads to the problem eluded in (7) where functions of constants of motion are not necessarily constants of motion—since, now, some function of the hamiltonian operator, say $f(H)$, might not even commute with H .

We might see this in a straightforward way if we look for the equations satisfied by the density functions $F(q, p; t)$. If we write $F^Q(q, p; t)$ for the function obtained from the quantum mechanical amplitudes, as above, and write $F^C(q, p; t)$ for the classical distribution, we have, by hypothesis

$$\frac{\partial F^C}{\partial t} + \{H, F^C\} = 0 \quad (49)$$

while, it is possible to show[16], the equation obeyed by function $F^Q(q, p; t)$ is given by

$$\frac{\partial F^Q}{\partial t} + \frac{2}{\hbar} \sin \frac{\hbar}{2} \left\{ \frac{\partial}{\partial p}, \frac{\partial}{\partial q} \right\} H(q, p) F^Q(q, p; t) = 0. \quad (50)$$

This makes clear our point of view. We began with equation (49) and end with another different equation (50). This is another argument for concluding that we shall not have

$$F^C(q, p; t) = F^Q(q, p; t). \quad (51)$$

In some special cases (the harmonic oscillator is an example) equation (49) is equivalent to equation (50). Even in such cases we shall not have (51) since we have postulated, in passing from the former to the later, one specific functional form for our densities and also a dispersion relation given by (31).

It is also important to stress that the ‘classical limit’ cannot be generally attained by taking the limit $\hbar \rightarrow 0$ as usually stated in the literature. This was also shown elsewhere[18].

The important result (50) means exactly what we have said above. Indeed, even if some quantity does commute with the Hamiltonian operator, some functional of it (e.g. its powers) will not generally satisfy this requirement and will not be a constant of motion[16]. This is another result that indicates that expression (50) must be only an approximation. This was already exemplified above in expressions (7) and (8), where we have noted that the square of the hamiltonian does not commute with the hamiltonian itself and cannot be a constant of motion.

This same result might be exemplified from Dirac’s rule. Since this rule uses precisely the substitution of the classical Poisson brackets by the commutator[3] and does not produce, in general, unambiguous operators, the one-to-one correspondence of classical and quantum-mechanical constants of motion generally fails.

6 Conclusions

Until now we have been concerned with reconstructing quantum mechanics from classical mechanics[7, 8, 9, 10, 11, 12, 13, 14]. This task was taken from the mathematical and epistemological points of view and there are, still, much work awaiting—for example, the reconstruction of quantum field theory.

This paper, however, has undertaken a different approach. The theme we have chosen to approach here lies in the frontier of applicability of the quantum formalism itself.

Indeed, one of the most fundamental axioms of the orthodox *interpretation* of quantum theory is the one-to-one correspondence between classical functions (to be quantized) and quantum operators (observables). We have thus avoided to contradict this postulate since it comes naturally from our own derivation of quantum mechanics.

We were then led to adopt Weyl's procedure, or the one we have derived from our own theory, as the correct one for the formation of quantum mechanical operators. The most striking consequence of our choice was to be left with energy eigenstates presenting energy dispersions.

We then have shown that the appearance of these dispersions are pertinent to the formalism since we derived them from the quantum phase-space distributions $F^Q(x, p; t)$, defined above, which keep the same physical content (they are only Fourier transforms of the quantum density function) of the quantum distributions.

The unavoidable appearance of energy eigenstates with energy dispersion points in the direction of an interpretation of these states very different from the one given by the orthodox epistemological approach. We must abandon, for example, the idea that when performing a measurement upon an energy eigenstate we do not disturb this energy value and the next measurement will observe the same value for this variable.

Concerning the formalism, the adoption of our procedure of operator formation leads to the physically unacceptable result according to which functions of constants of motion are not themselves constants of motion. It is interesting to note, however, that this pathological behavior comes from the attempt to map a commutative ring into a non-commutative one, and is of the order of, at least, the square of Planck's constant; since the magnitude of this error is highly beyond experimental precision, we may still expect this pathology not to be easily empirically detectable—this, of course, does not preclude us from studying the theme, since we are interested here in placing the quantum *theory* on the most solid ground. All these results might be also derived from the quasi-Liouville equation (50) obeyed by the quantum phase-space distributions above.

References

- [1] Agarwal, G. S. and Wolf, E., Phys. Rev. D **2**, 2161,2187,2206 (1970).
- [2] Groenewold, H. J., Physica **12**, 405 (1946).
- [3] Shewell, J. R., Am. J. Phys. **27**, 16 (1959).
- [4] Pool, J. T., J. Math. Phys. **7**, 66 (1966).
- [5] Metha, C. L. and Sudarshan, E. C. G., Phys. Rev. **138**, B274 (1965).
- [6] Misra, S. P. and Shankara, T. S., J. Math. Phys. **9**, 299 (1968).
- [7] Olavo, L. S. F., quant-ph/9503020
- [8] Olavo, L. S. F., quant-ph/9503021
- [9] Olavo, L. S. F., quant-ph/9503022
- [10] Olavo, L. S. F., quant-ph/9503024
- [11] Olavo, L. S. F., quant-ph/9503025
- [12] Olavo, L. S. F., quant-ph/9509012
- [13] Olavo, L. S. F., quant-ph/9509013
- [14] Olavo, L. S. F., quant-ph/9511028
- [15] Wigner, E. P., Phys. Rev. **40**, 749 (1932).
- [16] Moyal, J. E., Proc. Cambridge Phil. Soc. **45**, 99 (1949).
- [17] Schiff, L., " *Quantum Mechanics*", (McGraw-Hill Book Co., Singapore, 1965).
- [18] Cohn, J., Am. J. Phys. **40**, 463 (1972).
- [19] Ali, S. and Prugovecki, E., J. Math. Phys. **18**, 219 (1977).
- [20] Ali, S. and Prugovecki, E., Physica **89A**, 501 (1977).
- [21] Ali, S. and Prugovecki, E., Intern. J. of Theor. Phys. **16**, 689 (1977).
- [22] Prugovecki, E., Ann. of Phys. **110**, 102 (1978).
- [23] Prugovecki, E., Physica **91A**, 202 (1978).
- [24] Prugovecki, E., Physica **91A**, 229 (1978).
- [25] Prugovecki, E., J. Math. Phys. **19**, 2260 (1978).
- [26] Prugovecki, E., J. Math. Phys. **19**, 2271 (1978).

- [27] Prugovecki, E., Phys. Rev. D **18**, 3655 (1978).
- [28] Prugovecki, E., Found. of Phys. **9**, 575 (1979).
- [29] Ali, S., Gagnon, R. and Prugovecki, E., Can. J. Phys. **59**, 807 (1981).

n	energy	$\Delta q \Delta p$	ΔE
0	1/2	1/2	1/2
1	3/2	3/2	1/2
2	5/2	5/2	1/2
3	7/2	7/2	1/2
4	9/2	9/2	1/2
5	11/2	11/2	1/2
6	13/2	13/2	1/2
7	15/2	15/2	1/2

Table 1: Mean energies and dispersions for the harmonic oscillator derived from the quantum phase-space distributions.

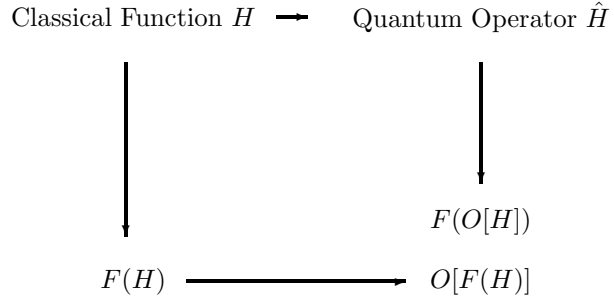


Figure 1: Graph showing the fail of von Neumann's process of quantization.

